First-Order Logic over Linear Type Theory

Reuben Hillyard

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Overview

Set-up

- A first-order theory $\mathbb{T}$;

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- Just as a definable set gives a functor $\mathbf{Mod}(\mathbb{T}) \to \mathbf{Set}$, we get a pseudofunctor $C(-) : \mathbf{Mod}(\mathbb{T}) \to \mathbf{Cat}$;
- We would like to produce a theory $\mathbb{T}'$ extending $\mathbb{T}$, with forgetful functor $U : \mathbf{Mod}(\mathbb{T}') \to \mathbf{Mod}(\mathbb{T})$, such that the fibre $U^{-1}(M)$ over a $\mathbb{T}$-model M is $C(M)$.

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Intuition

This fails because **Mod** cannot see that C is a category.

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Slogan

To encode objects in terms of sets: Yoneda lemma.

Yoneda Embedding

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We will represent an object X of C by its representable functor $C(-, X) : C^{\mathrm{op}} \to \mathbf{Set}$.

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- $\rho(x, f)$ is defined iff $\pi(x) = t(f)$;
- $\pi(\rho(x, f)) = s(f)$;
- $\rho(x, \mathrm{id}_{\pi(x)}) = x$;
- $\rho(x, g \circ f) = \rho(\rho(x, g), f)$;

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- $\exists \theta \in F \colon \forall x \in F \colon \exists! x' \in C_1 \colon \rho(\theta, x') = x$.

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- $\exists \theta \in F : \forall x \in F : \exists! x' \in C_1 : \rho(\theta, x') = x$.

This works – we get $U^{-1}(M) \cong C(M)$.

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Problem

This makes essential use of **Set**; so it only works for categories.
We could not do this for higher categories, R-modules, etc..
Mod$(\mathbb{T})$ is always an ordinary category.

Solution Idea

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- More specifically, replace $(\mathbf{Set}, \times, \{\star\})$ with a symmetric monoidal category of our choice.
- The eventual goal is an enriched logic.

Linear Type Theory

Braid-$\otimes$

$$\frac{\begin{array}{c}\Gamma, r : Y \otimes Z, x : X, \Delta \vdash e : E \\ \Theta \vdash s : X \qquad \Phi \vdash t : Y \qquad \Psi \vdash u : Z\end{array}}{\begin{array}{c}\Gamma, \Theta, \Phi, \Psi, \Delta \vdash \texttt{let } r \otimes x = \sigma(s \otimes (t \otimes u)) \texttt{ in } e \equiv \\ \texttt{let } y \otimes x' = \sigma(s \otimes t) \texttt{ in let } z \otimes x = \sigma(x' \otimes u) \texttt{ in } e[y \otimes z / r] : E\end{array}}$$

Linear Type Theory

$$\frac{}{\cdot\ \texttt{ctx}}\ \text{Empty-ctx} \qquad \frac{\Gamma\ \texttt{ctx} \qquad X\ \texttt{type}}{\Gamma, x : X\ \texttt{ctx}}\ \text{Extended-ctx} \qquad \frac{X\ \texttt{type}}{x : X \vdash x : X}\ \text{Variable}$$

Linear Type Theory

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For each sort symbol X,

$$\text{Sort}\;\frac{}{X\ \mathtt{type}}$$

Linear Type Theory

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Extended-ctx
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Variable
$$\frac{X\ \texttt{type}}{x : X \vdash x : X}$$

For each sort symbol X, and function symbol $f : (X_1, \ldots, X_n) \to Y$,

Sort
$$\frac{}{X\ \texttt{type}}$$

Function
$$\frac{\Gamma_1 \vdash s_1 : X_1 \qquad \cdots \qquad \Gamma_n \vdash s_n : X_n \qquad \mathrm{Shuf}([\Gamma_1; \ldots; \Gamma_n]; \Delta)}{\Delta \vdash f(s_1, \ldots, s_n) : Y}$$

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Empty-mtx
$$\frac{}{\bullet\ \mathtt{mtx}}$$

Extended-mtx
$$\frac{\boldsymbol{\Xi}\ \mathtt{mtx} \qquad \Gamma\ \mathtt{ctx}}{\boldsymbol{\Xi}; \Gamma\ \mathtt{mtx}}$$

Term-mtx
$$\frac{\Gamma_1\ \mathtt{ctx} \qquad \cdots \qquad \Gamma_n\ \mathtt{ctx} \qquad \Gamma_i \vdash s : Z}{\Gamma_1; \ldots; \Gamma_n \vdash \langle s \rangle_i : Z}\ (1 \leq i \leq n)$$

First-Order Logic

Truth-form

$$\frac{\mathbf{\Xi}\ \mathtt{mtx}}{\mathbf{\Xi} \vdash \top\ \mathtt{prop}}$$

Conjunction-form

$$\frac{\mathbf{\Xi} \vdash \phi\ \mathtt{prop} \qquad \mathbf{\Xi} \vdash \psi\ \mathtt{prop}}{\mathbf{\Xi} \vdash \phi \wedge \psi\ \mathtt{prop}}$$

Disjunction-form

$$\frac{\mathbf{\Xi}\ \mathtt{mtx} \qquad \mathbf{\Xi} \vdash \phi_j\ \mathtt{prop} \quad \text{for } j \in J}{\mathbf{\Xi} \vdash \bigvee_{j \in J} \phi_j\ \mathtt{prop}}\ (J \text{ a set})$$

Equality-form

$$\frac{\mathbf{\Xi} \vdash s : Z \qquad \mathbf{\Xi} \vdash t : Z}{\mathbf{\Xi} \vdash s \doteq_Z t\ \mathtt{prop}}$$

$\exists$-form

$$\frac{\mathbf{\Xi}; \Gamma; \mathbf{\Omega} \vdash \phi\ \mathtt{prop}}{\mathbf{\Xi}; \mathbf{\Omega} \vdash (\exists\Gamma)\phi\ \mathtt{prop}}$$

$\exists_{\mathtt{new}}$-form

$$\frac{\mathbf{\Xi}; (\Gamma, x : X, \Delta); \mathbf{\Omega} \vdash \phi\ \mathtt{prop}}{\mathbf{\Xi}; (\Gamma, \Delta); \mathbf{\Omega} \vdash (\exists_{\mathtt{new}}\ x : X)\phi\ \mathtt{prop}}$$

First-Order Theories

For each relation symbol $R(Z_1; \ldots; Z_n)$,

$$\text{Relation}\;\frac{\boldsymbol{\Xi} \vdash s_1 : Z_1 \qquad \cdots \qquad \boldsymbol{\Xi} \vdash s_n : Z_n}{\boldsymbol{\Xi} \vdash R(s_1; \ldots; s_n)\ \texttt{prop}}$$

First-Order Theories

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Theories can have axioms of the form $\Gamma_1; \ldots; \Gamma_n | \phi_1; \ldots; \phi_m \vdash \psi$.

Monoids and Groups

The theory $\mathbb{M}$ of a monoid has:

- one sort symbol G;
- function symbols $m : (G, G) \to G$; $e : () \to G$;
- axioms:

$$x : G, y : G, z : G \vdash m(m(x, y), z) \doteq m(x, m(y, z))$$

$$x : G \vdash m(x, e()) \doteq x \qquad x : G \vdash m(e(), x) \doteq x$$

Monoids and Groups

The theory $\mathbb{G}$ of a group has:

- one sort symbol G;
- function symbols $m : (G, G) \to G$; $e : () \to G$;
- axioms:

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$$x : G \vdash m(x, e()) \doteq x \qquad x : G \vdash m(e(), x) \doteq x$$

$$x : G; \cdot \vdash (\exists_{\mathtt{new}}\ y : G)\ m(x, y) \doteq e()$$

Toposes

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$$\mathbb{M}\text{-mod}(\mathcal{E}) \cong \mathbf{Mon}(\mathcal{E}) \qquad\qquad \mathbb{G}\text{-mod}(\mathcal{E}) \cong \mathbf{Grp}(\mathcal{E})$$

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- In particular,

$$\mathbb{M}\text{-mod}([C^{\mathrm{op}}, \mathbf{Set}]) \cong \mathbf{Mon}([C^{\mathrm{op}}, \mathbf{Set}]) \cong [C^{\mathrm{op}}, \mathbf{Mon}]$$

$$\mathbb{G}\text{-mod}([C^{\mathrm{op}}, \mathbf{Set}]) \cong \mathbf{Grp}([C^{\mathrm{op}}, \mathbf{Set}]) \cong [C^{\mathrm{op}}, \mathbf{Grp}]$$

Graded Sets

From a (commutative) monoid M,
we get a (symmetric) monoidal topos $\mathbf{Set}_{/M}$ with:

$$I = \begin{array}{c} 1 \\ \big\downarrow{\scriptstyle e} \\ M \end{array} \qquad \begin{array}{c} X \\ \big\downarrow{\scriptstyle \phi} \\ M \end{array} \otimes \begin{array}{c} Y \\ \big\downarrow{\scriptstyle \psi} \\ M \end{array} = \begin{array}{c} X \times Y \\ \big\downarrow{\scriptstyle \phi \times \psi} \\ M \times M \\ \big\downarrow{\scriptstyle m} \\ M \end{array}$$

Graded Sets

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$$\mathbb{M}\text{-mod}(\mathbf{Set}_{/M}) \cong \mathbf{Mon}_{/M} \qquad \mathbb{G}\text{-mod}(\mathbf{Set}_{/M}) \hookrightarrow \mathbf{Mon}_{/M}$$

$\mathbb{Z}_2$-Graded Sets

A familiar special case is $\mathbb{Z}_2$-graded sets with:

$$I = (\{\star\}, \varnothing)$$

$$(X_E, X_O) \otimes (Y_E, Y_O) = (X_E \times Y_E + X_O \times Y_O,\ X_E \times Y_O + X_O \times Y_E)$$

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Split Complex Numbers: $\mathbb{R}[j]$

Pairs of real numbers with:

$$1 = 1 + 0j$$

$$(x_E + x_O j) \cdot (y_E + y_O j) = (x_E y_E + x_O y_O) + (x_E y_O + x_O y_E) j$$

Complex Numbers and Dual Numbers

Complex Numbers: $\mathbb{R}[i]$

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Complex Numbers and Dual Numbers

Dual Numbers: $\mathbb{R}[\varepsilon]$

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$$\mathbb{M}\text{-mod}(\mathbf{Set}[\varepsilon]) \cong \{(G, X) \quad \text{for } G : \mathbf{Mon};\ X : G\text{-}G\text{-}\mathbf{BiMod}\}$$

$$\mathbb{C}\text{-mod}(\mathbf{Set}[\varepsilon]) \cong \{(G, X) \quad \text{for } G : \mathbf{CMon};\ X : G\text{-}\mathbf{Mod}\}$$

$$\mathbb{G}\text{-mod}(\mathbf{Set}[\varepsilon]) \cong \{(G, \varnothing) \quad \text{for } G : \mathbf{Grp}\}$$

$$\mathbb{O}\text{-mod}(\mathbf{Set}[\varepsilon]) \cong \{(G, X) \quad \text{for } G : \mathbb{O}\text{-mod};\ X : G\text{-}\mathbf{Mod}\}$$

Next Steps

- Proof system; $\exists_{\mathtt{new}}$.
- Semantics; Multi-toposes; Classification.
- (Bunched) Type Theory.
- Restore the enrichment.
- Relation to Quantales and Quantum Logic.