Towards Cubical Type Theory for Stable ∞-Categories

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Why care about Stable ∞-Categories?

	Topos	Pointed	Semi-Additive	Abelian/Stable
1	**Set**	$\mathbf{Set}_\bullet$	**CMon**	**Ab**
∞	**Space**	$\mathbf{Space}_\bullet$	$\mathbb{E}_\infty$-**Space**	**Spec**

Favourite stable ∞-category **Spec**:

- Abelian groups up to homotopy;
- Generalized cohomology theories;
- CW complexes with negative-dimensional cells;
- *Spectra are to spaces what linear algebra is to algebra.* (Malkiewich 2023)

Why use type theory for ∞-Categories?

- ∞-categories are hard because of coherence conditions.
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- Type theories compute (unless they have axioms).

Overview

Goal

Define a type theory for stable ∞-categories, that computes.

Strategy

- In (Riley, Finster and Licata 2021), they axiomatise biproducts, and derive stability.
- Define a type theory where biproducts are a theorem.
- Follow the same method to derive stability.

Pointedness

Definition (Zero Object (Lurie 2017, Definition 1.1.1.1))

A zero object is an object that is both initial and terminal.

Definition (Pointed Category (Lurie 2017, Definition 1.1.1.1))

A pointed category is a category with a zero object.

Example ($\mathbf{Set}_{\bullet}$)

The category of pointed sets has a zero object given by the one point space.

Pointedness and Enrichment

Remark (Enrichment from Pointedness)

Let $\mathcal{C}$ be pointed.
We can canonically enrich it over $\mathbf{Set}_\bullet$ by equipping $\mathcal{C}(X, Y)$ with

$$X \to \mathbf{0} \to Y$$

Remark (Pointedness from Enrichment)

Let $\mathcal{C}$ be a $\mathbf{Set}_\bullet$-category with a terminal object $\mathbf{0}$.
Then $\mathbf{0}$ is also initial, since, for $f : \mathbf{0} \to Z$, we have

$$f = f \circ \mathrm{id}_{\mathbf{0}} = f \circ 0 = 0$$

Semi-Additivity

Definition (Binary Biproduct)

A biproduct of two objects A and B is an object that is both the coproduct and the product of A and B.

Definition (Semi-Additive Category)

A semi-additive category is a category with a zero object and binary biproducts.

Example (**CMon**)

The category of commutative monoids has finite biproducts given by product of the underlying sets with pointwise operations.

Remark

In a semi-additive category, we can write morphisms between biproducts as block matrices, as we can in **Vec**.

Semi-Additivity and Enrichment

Remark (Enrichment from Semi-Additivity)

Let $\mathcal{C}$ be semi-additive.
We can canonically enrich it over **CMon** by equipping $\mathcal{C}(X, Y)$ with

$$0 \equiv X \to \mathbf{0} \to Y \qquad f + g \equiv X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y$$

Remark (Semi-Additivity from Enrichment)

Let $\mathcal{C}$ be a **CMon**-category with finite products.
As before, the terminal object is also initial.
For objects A and B, the product $A \oplus B$ is also the coproduct, with inclusions and copairing given by

$$\iota_1 \equiv (\mathrm{id}_A, 0) : A \to A \oplus B \qquad \iota_2 \equiv (0, \mathrm{id}_B) : B \to A \oplus B$$

$$[f, g] \equiv f \circ \pi_1 + g \circ \pi_2 : A \oplus B \to C$$

Commutativity of Limits and Colimits

Proposition (Pointedness and Commutativity)

A category $\mathcal{C}$ is pointed iff it has an initial object and a terminal object and the functors picking them out commute.
Two constants commute iff they coincide; so this is iff $\mathbf{1} \cong \mathbf{0}$.

Proposition (Semi-Additivity and Commutativity)

A category $\mathcal{C}$ is semi-additive iff it has finite coproducts and products and the functors computing them commute.
That is, iff $\mathbf{1} \cong \mathbf{0}$ and $(A + B) \times (C + D) \cong (A \times C) + (B \times D)$.

Definition (Stable Category)

A stable category is a category with finite colimits and limits and the functors computing them commute.

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Oh no!

Every stable category is trivial.

Finite Colimits and Limits in ∞-Categories

Let $\mathcal{C}$ be a pointed ∞-category.

Definition (Reduced Suspension and Loop Space)

For an object $Z \in \mathcal{C}$, the reduced suspension and loop space, when they exist, are given by

$$\begin{array}{ccc} Z & \longrightarrow & \mathbf{0} \\ \downarrow & & \downarrow \\ \mathbf{0} & \longrightarrow & \Sigma Z \end{array} \qquad\qquad \begin{array}{ccc} \Omega Z & \longrightarrow & \mathbf{0} \\ \downarrow & & \downarrow \\ \mathbf{0} & \longrightarrow & Z \end{array}$$

We have $\mathcal{C}(\Sigma X, Y) \cong \Omega(\mathcal{C}(X, Y)) \cong \mathcal{C}(X, \Omega Y)$; so $\Sigma \dashv \Omega$.
In $\mathbf{Set}_\bullet$, $\Omega Z \cong \mathbf{0}$.

Stability

Definition (Stable ∞-Category (Antolín-Camarena 2017))

A stable ∞-category is an ∞-category with finite limits and colimits that commute.

Stability $\sim$ Equivalence (Lurie 2017, Proposition 1.4.2.11)

A stable ∞-category is a pointed ∞-category with pushouts and pullbacks, where the $\Sigma \dashv \Omega$ adjunction is an adjoint equivalence. That is, where the unit and co-unit of the adjunction are equivalences.

Stability $\sim$ Enrichment (Hillyard 2025, Remarks 4.21 and 4.22)

An ∞-category is stable iff
It is **Spec**-enriched and has **Spec**-enriched finite limits and colimits.

Spec-enriched implies $\mathbb{E}_\infty$-**Space**-enriched.

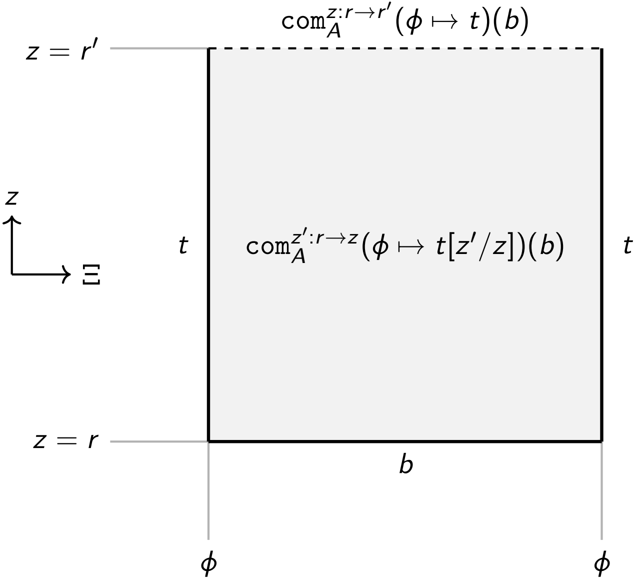
Why Cubical Type Theory?

- In HoTT, we have axioms `funExt` and `ua`.
- In Cubical Type Theory, we add new primitives from which these can be proven, restoring computation.
- Axiom S in (Riley, Finster and Licata 2021) resembles `ua`.

Cubical Type Theory

- Reify paths as maps out of a formal interval $\mathbb{I}$.
- Interval variables get their own section of the context.
- Interval term-formers: constants i0 and i1, and optionally the connectives $\wedge$, $\vee$, and $\neg$.
- Context section for formulas.
- Disjunction elimination and Kan composition.

Kan Composition



Stable Type Theory

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Stable Type Theory is cubical type theory with type-formers for finite limits and colimits, and $\mathbb{E}_\infty$-space structure for every type.

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We won't have dependent types.
Consider $p + p'$ for $p, p' : \sum_{a:A} B$.
Naïvely, $p + p' \equiv (\pi_1(p) + \pi_1(p'), \pi_2(p) + \pi_2(p'))$.
But $\pi_2(p) : B[\pi_1(p)/a]$ and $\pi_2(p') : B[\pi_1(p')/a]$;
So, this is ill-typed.

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A Better Way?

$$\textsc{Sum Term}\ \frac{\Delta \vdash \gamma : \Gamma \qquad \Delta \vdash \gamma' : \Gamma \qquad \Gamma \vdash A \ \mathtt{type} \qquad \Delta \vdash a : A[\gamma] \qquad \Delta \vdash a' : A[\gamma']}{\Delta \vdash a + a' : A[\gamma + \gamma']}$$

Enrichment

Every type has an $\mathbb{E}_\infty$-space structure:

- For a type A, we have $0 : A$;
- For terms $a, b : A$, we have $a + b : A$;
- We have paths λ, ρ, σ, α exhibiting that $+$ is unital, commutative and associative;
- We have higher paths forever exhibiting coherence of the lower paths.

Every operation respects this structure:

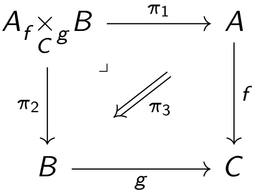
- For a map $f : A \to B$, we have δ_0, δ_+ and so on exhibiting that f preserves 0, $+$ and so on;
- And δ_{δ_0}...

These are rules, like Kan composition; not axioms.

Zero Type

- We have a type $\mathbf{0}$;
- We have a term $* : \mathbf{0}$;
- For any term $s : \mathbf{0}$, we have $s \equiv *$;
- These are the usual rules for a negative unit type;
- As before, we can prove that $\mathbf{0}$ is initial.

Pullback Types

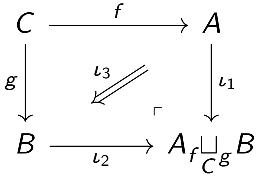


Behaves as $\sum_{a:A} \sum_{b:B} \mathrm{Path}_C(f(a), g(b))$.
Kan and enrichment operations are pointwise.

×-+

$$\frac{\Xi|\Phi|\Gamma \vdash p : A {}_f\times_C{}_g B \qquad \Xi|\Phi|\Gamma \vdash q : A {}_f\times_C{}_g B}{\begin{aligned}\Xi|\Phi|\Gamma \vdash p + q \equiv \big(&\pi_1(p) + \pi_1(q), \\ &\pi_2(p) + \pi_2(q), \\ &x \mapsto \mathtt{com}_C^{z:\mathrm{i0}\to\mathrm{i1}}(\\ &\qquad x = \mathrm{i0} \mapsto \delta_+^z(f; \pi_1(p), \pi_1(q)), \\ &\qquad x = \mathrm{i1} \mapsto \delta_+^z(g; \pi_2(p), \pi_2(q)), \\ &)(\pi_3^x(p) + \pi_3^x(q))\big) : A {}_f\times_C{}_g B\end{aligned}}$$

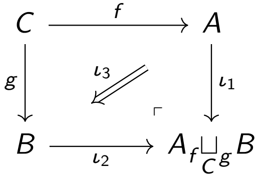
Pushout Types



Kan and enrichment operations are formal,
And the eliminator preserves them.

$$
\begin{aligned}
&\texttt{match}\ p\ \{\\
&\quad \iota_1(a) \mapsto s,\\
&\quad \iota_2(b) \mapsto t,\\
&\quad \iota_3^x(c) \mapsto u,\\
&\\
&\}
\end{aligned}
$$

Pushout Types



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&\quad p'\ \texttt{if}\ \phi \mapsto v,\\
&\}
\end{aligned}
$$

Special Cases

$$\Sigma A \equiv \mathbf{0}\,{}_0\underset{A}{\sqcup}{}_0\,\mathbf{0} \qquad \Omega A \equiv \mathbf{0}\,{}_0\underset{A}{\times}{}_0\,\mathbf{0}$$

$$A + B \equiv A\,{}_0\underset{\mathbf{0}}{\sqcup}{}_0\,B \qquad A \times B \equiv A\,{}_0\underset{\mathbf{0}}{\times}{}_0\,B$$

When we introduce or eliminate these, we can omit terms that we know must be zero.

Biproducts

For types A and B, we get a map $\psi : A + B \to A \times B$ given by

$$\psi(p) \equiv \texttt{match}\ p\ \{ \quad \iota_1(a) \mapsto (a, 0), \quad \iota_2(b) \mapsto (0, b), \quad \}$$

$$\begin{bmatrix} \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_B \end{bmatrix}$$

We want to show that this map is an equivalence.
Our proposed inverse is $\phi : A \times B \to A + B$ given by

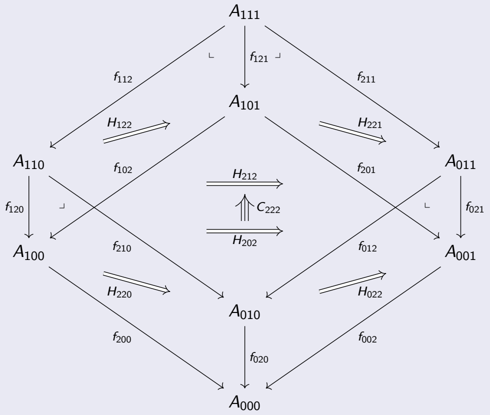
$$\phi(p) \equiv \iota_1(\pi_1(p)) + \iota_2(\pi_2(p))$$

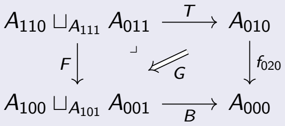
$$\begin{bmatrix} \mathrm{id}_A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{id}_B \end{bmatrix}$$

With the eliminator for $+$, we can prove that these are indeed mutually inverse.

Deriving Stability

Theorem (Cube of Pullbacks (Rijke 2019, Theorem 2.2.12))





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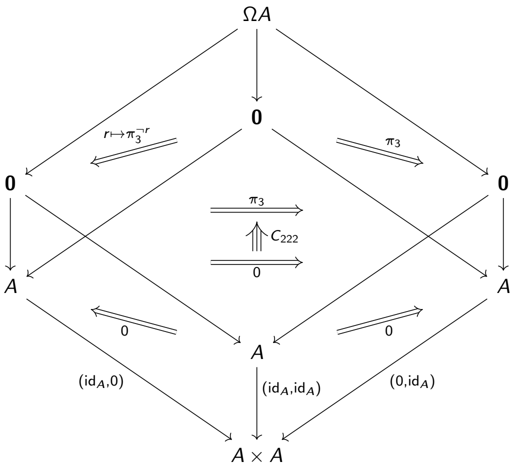

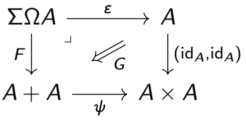

We have ψ an equivalence; so ε is an equivalence.

Apply Little Blakers-Massey Theorem to get η an equivalence.

Problem with Cube of Pullbacks

Non-Theorem!

While Cube of Pullbacks is provable in HoTT,
We can't prove it in StabTT with our eliminator for pushout.

Future Work

- Strengthen eliminator for pushout and prove the cube theorem.
- Prove Little Blakers-Massey Theorem to get η an equivalence.

Conjecture

Stable type theory is the internal language for stable ∞-categories.

- Add smash product, and Σ^∞ and Ω^∞.
- Simplify the higher coherators. ($A \simeq \Omega\Sigma A$)

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